JKAU: Sci., **vol. 17,** pp. 177-193 (2005 A.D. / 1425 A.H.)

Unified Symbolic Algorithm for Some Expansions of the Two-Body Problem Using Lagrange's Fundamental Invariants

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ABSTRACT. In this paper, four symbolic computing algorithms are developed for time series expansions of the following two-body problem: The Lagrange **f** and **g** functions, radial distance, the normalized inner product of the position and velocity vectors, and the eccentric anomaly of elliptic motion. To obtain these expansions once and for all, we proposed a unified approach in which Lagrange's fundamental invariants were used to develop time series solution of single harmonic oscillator equation.

1. Introduction

It is undoubtly true that, the analytical formulae of space dynamics usually offer much deeper insight into the nature of the problems to which they refer. Moreover, the nowadays existing symbols used for manipulating digital computer programs, opened the gate towards establishing new branch of space dynamics known as the algorithmization of space dynamics [Brumberg, 1995]. A great effort has been devoted up to now, and is being devoted at present to develop symbolic computing algorithms for some problems of space dynamics [*e.g.* Brumberg, 1995; Sharaf and Saad, 1997; Sharaf, *et al.*, 1998; Vinti, 1998; Sharaf and Banajah, 2001].

Coping with this important line of recent approach, the present paper is devoted to establish four symbolic computing algorithms for time series expansions of the following two-body problems: The Lagrange **f** and **g** functions, radial distance, the normalized inner product (denoted by **σ**) of the position and velocity vectors, and finally, the eccentric anomaly of elliptic orbits. The first three expansions are universal in the sense that, they could be used for any of the conic orbits (elliptic, parabolic, or hyperbolic) while the last expansion is used only for elliptic orbits.

To obtain these expansions once and for all, we proposed a unified approach, in which Lagrange's fundamental invariants were used to develop time series solution of harmonic oscillater equation of the form $\ddot{q}+\in q = 0$.

2. Basic Formulations

2.1. Two-Body Formulations

● The equation describing the relative motion of the two bodies of masses m_1 and m_2 and in rectangular coordinates is

$$\frac{d}{dt}\vec{v} = \ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r} \ , \tag{2.1}$$

where **μ** is the gravitational parameter (universal gravitational constant times the sum of the two masses) $\vec{\mathbf{r}}$ and $\vec{\mathbf{v}}$ are the position and velocity vectors given in components as

$$\vec{r} = x\,\vec{i}_x + y\,\vec{i}_y + z\,\vec{i}_z \ , \tag{2.2}$$

$$\vec{v} = \dot{x}\,\vec{i}_x + \dot{y}\,\vec{i}_y + \dot{z}\,\vec{i}_z \ , \tag{2.3}$$

$\vec{i}_x$, $\vec{i}_y$, and $\vec{i}_z$ are the unit vectors along the coordinate axes x, y and z respectively and

$$r = \left(x^2 + y^2 + z^2\right)^{1/2} \ . \tag{2.4}$$

Equation (2.1) is unchanged if we replace $\vec{r}$ with $-\vec{r}$. Thus Equation (2.1) gives the motion of the body of mass m_2 relative to the body of the mass m_1, or the motion of m_1 relative to m_2. Also if we replace **t** with **–t**, Equations (2.1) unchanged.

● At any time, $\vec{r}$ and $\vec{v}$ can be expressed as

$$\vec{r} = L\ \vec{i}_e + T\ \vec{i}_p \ , \tag{2.5}$$

$$\vec{v} = \dot{L}\ \vec{i}_e + \dot{T}\ \vec{i}_p \ , \tag{2.6}$$

where **(L, T)** are the pericenter coordinates of one of the bodies in its orbit about the other body and ($\dot{\mathbf{L}}$, $\dot{\mathbf{T}}$) are their time derivatives. These coordinates are of different forms for the different types (elliptic, parabolic, hyperbolic) of the two body motion [Danby, 1988] and are not needed to be specified here. The unit vectors $\vec{\mathbf{i}}_{\mathbf{e}}$, $\vec{\mathbf{i}}_{\mathbf{p}}$ and $\vec{\mathbf{i}}_{\mathbf{h}}$ are selected such that, $\vec{\mathbf{i}}_{\mathbf{e}}$ and $\vec{\mathbf{i}}_{\mathbf{p}}$ in the body's own orbital plane with $\vec{\mathbf{i}}_{\mathbf{e}}$ in the direction of pericenter, while $\vec{\mathbf{i}}_{\mathbf{p}}$ and $\vec{\mathbf{i}}_{\mathbf{h}}$ are chosen to make the coordinate system right-handed.

● Among the integrals of the two-body problem are the conservation of angular momentum vector $\vec{\mathbf{h}}$ where,

$$\vec{h} = \sqrt{\mu p}\ \vec{i}_h = \sqrt{\mu p}\ \left(\vec{i}_e \times \vec{i}_p\right) = \vec{r} \times \vec{v} \tag{2.7}$$

and the energy integral

$$\upsilon^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right), \tag{2.8}$$

where **p** and **a** are respectively, the semi-latus rectum and the semi-major axis of the orbit. From Equations (2.5), (2.6) and (2.7) we get

$$L\dot{T} - \dot{L}T = \sqrt{\mu p} \tag{2.9}$$

2.2. Lagrange's Fundamental Invariants

Lagrange's fundamental invariants [Battin,1999] $\in$, λ and ψ are defined as

$$\in = \mu / r^3 \quad , \tag{2.10.1}$$

$$\lambda = \frac{1}{r^2}\langle \vec{r}\ ,\ \vec{v}\rangle , \tag{2.10.2}$$

$$\psi = \frac{1}{r^2}\langle \vec{v}\ ,\ \vec{v}\rangle , \tag{2.10.3}$$

where $\langle \vec{\mathbf{A}}, \vec{\mathbf{B}}\rangle$ is used to denote the scalar product of two the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. The quantities $\in$, λ and ψ are "invariant" because they are independent of the selected coordinate system and "fundamental" because they form a closed set under the operation of time derivative, where

$$\frac{d\in}{dt} = -3\in\ \lambda\ , \tag{2.11.1}$$

$$\frac{d\lambda}{dt} = \psi - \in -2\lambda^2\ , \tag{2.11.2}$$

$$\frac{d\psi}{dt} = -2\lambda(\in + \psi)\ . \tag{2.11.3}$$

3. Solution by Power Series

The basic differential equations that concerns us in the subsequent analysis are

$$\frac{d^2q}{dt^2}+\in q=0\ , \tag{3.1}$$

Together with Equations (2.6) written as

$$\frac{d\in}{dt}+3\in\lambda=0, \tag{3.2}$$

$$\frac{d\lambda}{dt}+\in+2\lambda^2-\psi=0\ , \tag{3.3}$$

$$\frac{d\psi}{dt}+2\lambda(\in+\psi)=0,) \tag{3.4}$$

Where **$\in$**, **λ** and **ψ** are defined by Equations (2.10). Power series solutions for the above set of differential equations could be developed as follows [Battin,1999].

Expand each of the functions **q, $\in$** , **λ** and **ψ** in a Taylor's series in time

$$q=\sum_{n=0}^{\infty}q_n(t-t_0)^n\ , \tag{3.5}$$

$$\in=\sum_{n=0}^{\infty}\in_n(t-t_0)^n\ , \tag{3.6}$$

$$\lambda=\sum_{n=0}^{\infty}\lambda_n(t-t_0)^n\ . \tag{3.7}$$

$$\psi=\sum_{n=0}^{\infty}\psi_n(t-t_0)^n\ . \tag{3.8}$$

The procedure now is to substitute the four series given by Equations (3.5)-(3.8) into the four differential Equations (3.1)-(3.4) and then solve for the coefficients **q_n**, **$\in_n$**, **λ_n** and **ψ_n** by comparison of the coefficients of power of time. The central mathematical device used is the general relation

$$\left(\sum_{n=0}^{\infty}\alpha_n x^n\right)\left(\sum_{n=0}^{\infty}\beta_n x^n\right)=\sum_{n=0}^{\infty}\ \sum_{v=0}^{n}\ \alpha_v\beta_{n-v}\ \ x^n\ , \tag{3.9}$$

which converts the product of two infinite series to a double summation. The resulting recurrence relations are

$$q_{n+2} = \frac{-1}{(N+1)\,(n+2)} \sum_{i=0}^{n} \in_i q_{n-1}, \tag{3.10}$$

$$\in_{n+1} = \frac{-3}{n+1} \sum_{i=0}^{n} \in_i \lambda_{n-i} , \tag{3.11}$$

$$\lambda_{n+1} = \frac{1}{n+1}\left\{\psi_n - \in_n - 2 \sum_{i=0}^{n} \lambda_i \lambda_{n-i}\right\}, \tag{3.12}$$

$$\psi_{n+1} = \frac{-2}{n+1} \sum_{i=0}^{n} \lambda_i \left(\in_{n-i} + \psi_{n-i}\right) . \tag{3.13}$$

The starting values for these recurrence relations are $\mathbf{q}_0 \equiv \mathbf{q}(t_0)$; $\mathbf{q}_1 = \dot{\mathbf{q}}(\mathbf{t}_0)$, $\boldsymbol{\in}_0 \equiv \boldsymbol{\in}(\mathbf{t}_0)$, $\boldsymbol{\lambda}_0 \equiv \boldsymbol{\lambda}(\mathbf{t}_0)$ and $\boldsymbol{\psi}_0 \equiv \psi(\mathbf{t}_0)$. The first two values $\mathbf{q}_0$ and $\mathbf{q}_1$ being known for any given problem, while the other starting values $\boldsymbol{\in}_0$, $\boldsymbol{\lambda}_0$ and $\boldsymbol{\psi}_0$ could be computed from the initial values $\vec{\mathbf{r}}_0 \equiv \vec{\mathbf{r}}(\mathbf{t}_0)$ and $\vec{\mathbf{v}}_0 \equiv \vec{\mathbf{v}}(\mathbf{t}_0)$, so we get from Equations (2.5) the values

$$\in_0 = \mu / r_0^3 , \tag{3.14}$$

$$\lambda_0 = \left(x_0\, \dot{x}_0 + y_0\, \dot{y}_0 + z_0\, \dot{z}_0\right) / r_0^2 , \tag{3.15}$$

$$\psi_0 = \left(\dot{x}_0^2 + \dot{y}_0^2 + \dot{z}_0^2\right) / r_0^2 , \tag{3.16}$$

where

$$r_0 = \left(x_0^2 + y_0^2 + z_0^2\right)^{½} . \tag{3.17}$$

Using the symbolic manipulation capability of the software package Mathematica, we generate the coefficients $\mathbf{q_j}$; $\mathbf{j}$ = 2,3, ... 10, in terms of the known initial values and and are listed in Table 1.

In the following sections, applications of the above formulations will be considered.

TABLE 1. Symbolic expresions of the g_j coefficients ; $j = 2, 3, \ldots 10$.

$$g_2 = -\frac{1}{2} q_0 \in_0$$

$$g_3 = -\frac{1}{6} \in_0 \left(q_1 - 3q_0\lambda_0\right)$$

$$g4 = -\frac{1}{24} \in_0 \left(6q_1\lambda_0 + q_0\left(-2\in_0 + 3\left(-5\lambda_0^2 + \psi_0\right)\right)\right)$$

$$g5 = -\frac{1}{120} \in_0 \left(15q_0\lambda_0\left(2\in_0 + 7\lambda_0^2 - 3\psi_0\right) + q_1\left(-8\in_0 + 9\left(-5\lambda_0^2 + \psi_0\right)\right)\right)$$

$$g_6 = \frac{1}{720} \in_0$$
$$\left(30q_1\lambda_0\left(5\in_0 + 14\lambda_0^2 - 6\psi_0\right) - q_0\left(22\in_0^2 + 6\in_0\left(70\lambda_0^2 - 11\psi_0\right) + 45\left(21\lambda_0^4 - 14\lambda_0^2\psi_0 + \psi_0^2\right)\right)\right)$$

$$g_7 = \frac{1}{5040}\left(\in_0\left(-q_1\left(172\in_0^2 + 36\in_0\left(70\gamma_0^2 - 11\psi_0\right) + 225\left(21\lambda_0^4 - 14\lambda_0^2\psi_0 + \psi_0^2\right)\right) + \right.\right.$$
$$\left.\left. 63q_0\lambda_0\left(12\in_0^2 + 4\in_0\left(25\lambda_0^2 - 9\psi_0\right) + 5\left(33\lambda_0^4 - 30\lambda_0^2\lambda_0 + 5\psi_0^2\right)\right)\right)\right)$$

$$g_8 = \frac{1}{40320}\left(\in_0\left(126q_1\lambda_0\left(52\in_0^2 + 14\in_0\left(25\lambda_0^2 - 9\psi_0\right) + 15\left(33\lambda_0^4 - 30\lambda_0^2\psi_0 + 5\psi_0^2\right)\right) - \right.\right.$$
$$q_0\left(584\in_0^3 + 36\in_0^2\left(560\lambda_0^2 - 73\psi_0\right) + 54\in_0\left(1925\lambda_0^4 - 1120\lambda_0^2\psi_0 + 67\psi_0^2\right) + \right.$$
$$\left.\left.\left. 315\left(429\lambda_0^6 - 495\lambda_0^4\psi_0 + 135\lambda_0^2\psi_0^2 - 5\psi_0^3\right)\right)\right)\right)$$

$$q_9 = \frac{1}{362880}$$
$$\left(\in_0\left(15q_0\lambda_0\left(2368\in_0^3 + 444\in_0^2\left(77\lambda_0^2 - 24\psi_0\right) + 18\in_0\left(7007\lambda_0^4 - 5698\lambda_0^2\psi_0 + 827\psi_0^2\right) + \right.\right.\right.$$
$$\left. 189\left(715\lambda_0^6 - 1001\lambda_0^4\psi_0 + 385\lambda_0^2\psi_0^2 - 35\psi_0^3\right)\right) - q_1\left(7136\in_0^3 + 108\in_0^2\left(1785\lambda_0^2 - 232\psi_0\right) + \right.$$
$$\left.\left.\left. 432\in_0\left(1925\lambda_0^4 - 1120\lambda_0^2\psi_0 + 67\psi_0^2\right) + 2205\left(429\lambda_0^6 - 495\lambda_0^4\lambda_0 + 135\lambda_0^2\psi_0^2 - 5\psi_0^3\right)\right)\right)\right)$$

$$g_{10} = \frac{1}{3628800}$$
$$\left(\in_0\left(30q_1\lambda_0\left(15220\in_0^3 + 12\in_0^2\left(14938\lambda_0^2 - 4647\psi_0\right) + 81\in_0\left(7007\lambda_0^4 - 5698\lambda_0^2\psi_0 + 827\right.\right.\right.\right.$$
$$\left.\left.\psi_0^2\right) + 756\left(715\psi_0^6 - 1001\lambda_0^4\psi_0 + 385\lambda_0^2\psi_0^2 - 35\psi_0^3\right)\right) - q_0\left(28384\in_0^4 + \right.$$
$$48\in_0^3\left(31735\psi_0^2 - 3548\psi_0\right) + 54\in_0^2\left(245245\lambda_0^4 - 126940\lambda_0^2\psi_0 + 6559\psi_0^2\right) +$$
$$90\in_0\left(420420\lambda_0^6 - 441441\lambda_0^4\psi_0 + 107514\lambda_0^2\psi_0^2 - 3461\psi_0^3\right) +$$
$$\left.\left.\left. 14175\left(2431\psi_0^8 - 4004\lambda_0^6\psi_0 + 2002\lambda_0^4\psi_0^2 - 308\lambda_0^2\psi_0^3 + 7\psi_0^4\right)\right)\right)\right)$$

4. Symbolic Computation of Lagrange f and g Functions

4.1. Definition

It is well known that, the Taylor series expansion of the coordinates of a body in Keplerian motion about the attracting mass can be written as

$$\vec{r} = \dot{f}\ \vec{r}_0 + \dot{g}\ \vec{v}\ , \tag{4.1}$$

where $\vec{\mathbf{r}}$ is the position vector at time $\mathbf{t}$, while $\vec{\mathbf{r}}_0$ and $\vec{\mathbf{v}}_0$ are the position and velocity vectors at the initial time $\mathbf{t}_0$. The coefficients **f** and **g** are functions of $\mathbf{t}$-$\mathbf{t}_0$ and are called *Lagrange* **f** *and* **g** *functions*.

From Equation (4.1) we get

$$\vec{v} = \dot{f}\ \vec{r}_0 + \dot{g}\ \vec{v}\ , \tag{4.2}$$

where $\vec{\mathbf{v}}$ is the velocity vector at time $\mathbf{t}$.

4.2. Functional Relations

Lagrange functions **f** and **g** and their time derivatives satisfy some functional relations are following

1 – For any two point $\mathbf{r}$ and $\mathbf{r}_0$ on any type of the two-body motion at the times $\vec{\mathbf{t}}$ and $\mathbf{t}_0$, then

$$f\ \dot{g} - g\ \dot{f}\ = 1\ . \tag{4.3}$$

This relation can be proved using the conservation of the angular momentum [Equation (2.7)] at the two epochs, so $\vec{\mathbf{r}} \times \vec{\mathbf{v}} = \vec{\mathbf{r}}_0 \times \vec{\mathbf{v}}_0$, then using Equations (4.1) and (4.2) for the left hand side.

Equation (4.3) implies that, given any three of the four functions or $\mathbf{f}$, $\mathbf{g}$, $\dot{\mathbf{f}}$ or $\dot{\mathbf{g}}$, we can solve for the remaining ones.

2 – The functions $\mathbf{f}$, $\mathbf{g}$, $\dot{\mathbf{f}}$ and $\dot{\mathbf{g}}$, and could be expressed in terms of the pericenter coordinates $(\mathbf{L}, \mathbf{T})$ and their time derivatives $(\dot{\mathbf{L}}, \dot{\mathbf{T}})$ as follows.

● Let $(\mathbf{L}, \mathbf{T}, \dot{\mathbf{L}}, \dot{\mathbf{T}})$ without subscript are the values of these quantities at time $\mathbf{t}$, $(\mathbf{L}_0, \mathbf{T}_0, \dot{\mathbf{L}}_0, \dot{\mathbf{T}}_0)$ are the corresponding quantities at time $\mathbf{t}_0$. Then solving Equations (2.5) and (2.6) simultaneously for $\vec{\mathbf{i}}_e$ and $\vec{\mathbf{i}}_p$ at time $\mathbf{t}_0$ we get

$$\vec{i} = \frac{1}{\sqrt{\mu p}}\left(\dot{T}_0\vec{r}_0 - T_0\vec{v}_0\right)\ ;\ \vec{i}_p = \frac{1}{\sqrt{\mu p}}\left(-\dot{L}_0\vec{r}_0 + L_0\vec{v}_0\right)\ .$$

● Using these two expressions of the unit vectors $\vec{\mathbf{i}}_e$ and $\vec{\mathbf{i}}_p$ into Equations (2.5) and (2.6) for $\vec{\mathbf{r}}$ and $\vec{\mathbf{v}}$ at time $\mathbf{t}$ and then comparing the coefficients of $\vec{\mathbf{r}}_0$ and $\vec{\mathbf{v}}_0$ of the resulting equations with those of Equations (4.1) and (4.2), we get

$$f = \frac{1}{\sqrt{\mu p}}\left(L\dot{T}_0 - T\dot{L}_0\right) \; ; \; g = \frac{1}{\sqrt{\mu p}}\left(-LT_0 + TL_0\right) , \tag{4.4}$$

$$\dot{f} = \frac{1}{\sqrt{\mu p}}\left(\dot{L}\dot{T}_0 - \dot{T}\dot{L}_0\right) \; ; \; \dot{g} = \frac{1}{\mu p}\left(-\dot{L}T_0 + \dot{T}L_0\right) . \tag{4.5}$$

3 – For any three points $\vec{\mathbf{r}}_0$, $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$ on any type of the two-body motion at the three times $\mathbf{t}_0$, $\mathbf{t}_1$ and $\mathbf{t}_2$, then

$$f_{20} = f_{21} f_{10} + g_{21} \dot{f}_{10} , \tag{4.6.1}$$

$$g_{20} = f_{21} g_{10} + g_{21} \dot{g}_{10} , \tag{4.6.2}$$

$$\dot{f}_{20} = \dot{f}_{21} f_{10} + \dot{g}_{21} \dot{f}_{10} , \tag{4.6.3}$$

$$\dot{g}_{20} = \dot{f}_{21} g_{10} + \dot{g}_{21} \dot{g}_{10} , \tag{4.6.4}$$

where for example, $\mathbf{f}_{21}$ is Largrange $\mathbf{f}$ function expressed as a function of $\mathbf{t}_2$-$\mathbf{t}_1$. By using Equations (4.4) and (4.5) into the right hand sides of Equations (4.6) and using Equation (2.11) for $\mathbf{L\dot{T}} - \mathbf{\dot{L}T}$ in the resulting equations, the left hand sides of Equations (4.6) follow directly. Equations (4.6) could be written in matrix form as

$$\Phi_{20} = \Phi_{21}\Phi_{10} , \tag{4.7}$$

where the matrix $\Phi_{\ell k}$ is called the *transition matrix* to extrapolate $\vec{\mathbf{r}}_\ell$, $\vec{\mathbf{v}}_\ell$ at time $\mathbf{t}_\ell$ from $\vec{\mathbf{r}}_k$, $\vec{\mathbf{v}}_k$ at time $\mathbf{t}_k$ and is given as

$$\Phi_{\ell k} = \begin{vmatrix} f_{\ell k} & g_{\ell k} \\ \dot{f}_{\ell k} & \dot{g}_{\ell k} \end{vmatrix} . \tag{4.8}$$

The property of Equation (4.7) is very useful when the time difference $\mathbf{t}_2$-$\mathbf{t}_0$ is sufficiently large, we may have to repeat the process by decrementing the time interval several times. The transition matrices thus sequentially generated are, of course, multiplied together to produce the final desired transition matrix.

4 – From Equation (4.1) we get

$$\ddot{\vec{r}} = \ddot{f} \; \vec{r}_0 + \ddot{g} \; \vec{v}_0 .$$

Using this equation and Equation (4.1) into Equation (2.1) we obtain

$$\left(\ddot{f} + \in f\right) \vec{r}_0 + \left(\ddot{g} + \in g\right) \vec{v}_0 = 0 ,$$

that is the functions **f** and **g** each satisfy the differential Equation

$$\frac{d^2q}{dt^2} + \in q = 0 \,, \tag{4.9}$$

with the initial conditions

$$q\left(t_0\right) = \begin{cases} 1 & \text{for } q = f \\ 0 & \text{for } q = g \end{cases} \;; \qquad \dot{q}\left(t_0\right) = \begin{cases} 0 & \text{for } q = f \\ 1 & \text{for } q = g \end{cases} \;; \tag{4.10}$$

4.3. Symbolic Expressions for f and g Functions

Since Equation (4.9) is exactly the same as Equation (3.1), then according to Equation (3.5), **f** and **g** could be represented as power series in (**t-t**$_0$) as

$$f = \sum_{j=0}^{\infty} f_j (t - t_0 \tag{4.11}$$

$$g = \sum^{\infty} g_j (t - t_0 \tag{4.12}$$

where the starting values of the recursion algorithm for the coefficients are those given by Equations (3.14),(3.15) and (3.16) together with (see Equation (4.10))

$$f_0 = 1, \;\; f_1 = 0; \;\; g_0 = 1, \;\; g_1 \tag{4.13}$$

With above starting values, the general procedure of Section 3 yields for the coefficients $\mathbf{f_j}$ and $\mathbf{g_j}$, j = 0,1, ... 10 the symbolic expressions listed respectively in Tables 2 and 3.

Ten of the $\mathbf{f_j}$ and $\mathbf{g_j}$ coefficients are sufficient for any application, this is because, the principle application of Equations (4.1) and (4.2) has been in orbit determination problems where the time interval (**t-t**$_0$) is small. On the other hand, for the case in which the time interval is large we can use the process of repeat decremention of the time interval several times as mentioned after Equation (4.8), so each time interval could be made as small as we desired.

Finally, it should be mentioned that, the usage of the above algorithm of the **f** and **g** series for the initial value problem (the determination of $\vec{\mathbf{r}}$ and $\vec{\mathbf{v}}$ at time **t** from $\vec{\mathbf{r}}_0$ and $\vec{\mathbf{v}}_0$ at preceding time $\mathbf{t}_0$ from Equation (4.1) and(4.2)) is efficient due to some factors, Such as :

- Its recurrent nature facilitates the computations of any number of the coefficients needed for accurate predictions of $\vec{\mathbf{r}}$ and $\vec{\mathbf{v}}$.

TABLE 2. Symbolic expresions of the f_j coefficients ; j = 0, 1, ... 10.

$$f_0 = 1$$

$$f_1 = 0$$

$$f_2 = -\frac{\in_0}{2}$$

$$f_3 = \frac{\in_0 \lambda_0}{2}$$

$$f_4 = -\frac{1}{24} \in_0 \left(2 \in_0 + 15\lambda_0^2 - 3\psi_0\right)$$

$$f_5 = \frac{1}{8} \in_0 \lambda_0\left(2 \in_0 + 7\lambda_0^2 - 3\psi_0\right)$$

$$f_6 = -\frac{1}{720} \in_0 \left(22 \in_0^2 + 6 \in_0 \left(70\lambda_0^2 - 11\psi_0\right) + 45\left(21\lambda_0^4 - 14\lambda_0^2\psi_0 + \psi_0^2\right)\right)$$

$$f_7 = \frac{1}{80} \in_0 \lambda_0\left(12 \in_0^2 \; 4 \in_0 \left(25\lambda_0^2 - 9\psi_0\right) + 5\left(33\lambda_0^4 - 30\lambda_0^2\psi_0 + 5\psi_0^2\right)\right)$$

$$f_8 = -\frac{1}{40320}\left(\in_0 \left(584 \in_0^3 + 36 \in_3^2 \left(560\lambda_0^2 - 73\psi_0\right) + \right.\right.$$

$$\left.\left. 54 \in_0 \left(1925\lambda_0^4 - 1120\lambda_0^2\psi_0 + 67\psi_0^2\right) + 315\left(429\lambda_0^6 - 495\lambda_0^4\psi_0 + 135\lambda_0^2 - 5\psi_0^3\right)\right)\right)$$

$$f_9 = \frac{1}{24192}\left(\in_0\lambda_0\left(2368 \in_0^3 + 444 \in_0^2\left(77\lambda_0^2 - 24\psi_0\right) + 18 \in_0 \left(7007\lambda_0^4 - 5698\lambda_0^2\psi_0 + 827\right) + \right.\right.$$

$$\left.\left. 189\left(715\lambda_0^6 - 1001\lambda_0^4\psi_0 + 385\lambda_0^2\psi_0^2 - 35\psi_0^3\right)\right)\right)$$

$$f_{10} = -\frac{1}{3628800}$$

$$\left(\in_0 \left(28384 \in_0^4 + 48 \in_0^3\left(31735\lambda_0^2 - 3548\psi_0\right) + 54 \in_0^2 \left(245245\lambda_0^4 - 126940\lambda_0^2\psi_0 + 6559\psi_0^2\right) + \right.\right.$$

$$90 \in_0 \left(420420\lambda_0^6 - 441441\lambda_0^4\psi_0 + 107514\lambda_0^2\psi_0^2 - 3461\psi_0^3\right) +$$

$$\left.\left. 14175\left(2431\lambda_0^8 - 4004\lambda_0^6\psi_0 + 2002\lambda_0^4\psi_0^2 - 308\lambda_0^2\psi_0^3 + 7\psi_0^4\right)\right)\right)$$

TABLE 3. Symbolic expresions of the g_j coefficients ; $j = 0, 1,2, \ldots 10$.

$$g_0 = 0$$

$$g_1 = 1$$

$$g_2 = 0$$

$$g_3 = -\frac{\in_0}{6}$$

$$g_4 = \frac{\in_0 \lambda_0}{4}$$

$$g_5 = -\frac{1}{120} \in_0 \left(8 \in_0 + 45\lambda_0^2 - 9\psi_0\right)$$

$$g_6 = \frac{1}{24} \in_0 \lambda_0\left(5 \in_0 + 14\lambda_0^2 - 6\psi_0\right)$$

$$g_7 = -\frac{\in_0 \left(172 \in_0^2 + 36 \in_0 \left(70\lambda_0^2 - 11\psi_0\right) + 225\left(21\lambda_0^4 - 14\lambda_0^2 + \psi_0^2\right)\right)}{5040}$$

$$g_8 = \frac{1}{320} \in_0 \lambda_0\left(52 \in_0^2 + 14 \in_0 \left(25\lambda_0^2 - 9\psi_0\right) + 15\left(33\lambda_0^4 - 30\lambda_0^2\psi_0 + 5\psi_0^2\right)\right)$$

$$g_9 = -\frac{1}{362880}$$

$$\left(\in_0 \left(7136 \in_0^3 + 108 \in_0^2 \left(1785\lambda_0^2 - 232\psi_0\right) + 432 \in_0 \left(1925\lambda_0^4 - 1120\lambda_0^2\psi_0 + 67\psi_0^2\right) + \right.\right.$$

$$\left.\left.2205\left(429\lambda_0^6 - 495\lambda_0^4\psi_0 + 135\lambda_0^2\psi_0^2 - 5\psi_0^3\right)\right)\right)$$

$$g_{10} = \frac{1}{120960}$$

$$\left(\in_0 \lambda_0\left(15220 \in_0^3 + 12 \in_0^2 \left(14938\lambda_0^2 - 4647\psi_0\right) + 81 \in_0 \left(7007\lambda_0^4 - 5698\lambda_0^2\psi_0 + 827\psi_0^2\right) + \right.\right.$$

$$\left.\left.756\left(756\lambda_0^6 - 1001\lambda_0^4\psi_0 + 385\lambda_0^2\psi_0^2 - 35\psi_6^3\right)\right)\right)$$

● The solution of $\vec{\mathbf{r}}$ and $\vec{\mathbf{v}}$ does not need the solution of Kepler's equation and its variants for parabolic and hyperbolic orbits.

● The algorithm is universal in the sense that it could be used for any of the conic orbits.

● As we mentioned above, the algorithm could be used whatever the length of the time interval ($\mathbf{t}$-$\mathbf{t}_0$) may be.

5. Other Applications

5.1. Symbolic Computation of the Radial Distance

Radial distances are vital to a class of orbit determination problems which depend on range measurements [Escobal, 1976] .

The polar equation of the relative motion of the two-body problem is given as

$$\ddot{r} - r\dot{\theta}^2 = -\mu/r^2 \ , \tag{5.1}$$

where θ the true anomaly and $\dot{\theta} = \sqrt{\mu p}/r^2$, therefore Equation (5.1) could be written as

$$\ddot{r} + \frac{\mu}{r^3}(r - p) = 0 \ . \tag{5.2}$$

Let

$$q = r - p \ , \tag{5.3}$$

since **p** is constant for the two body problem, then from Equation (5.2), it is clear that **q** (Equation (5.3)) satisfies the differential Equation (3.1). Then by using the technique of Section 3 and Equation (5.3) we can develop power series expansion for the radial distance **r** as

$$r = p + \sum_{n=0}^{\infty} q_n (t - t)^n \ ,$$

where **q**'s are computed from Equations (3.10) to (3.13) with the initial conditions (3.14) to (3.17). It remains for the present algorithm to determine $\mathbf{q}_0$ and $\mathbf{q}_1$. Since

$$q_0 = r_0 - p = r_0 - h^2/u \ ; \ q_1 = \left.\frac{dq}{dt}\right|_{t=t_0} = \dot{r}_0 \ .$$

then, the power series expansion for becomes

$$r = r_0 + \sum_{n=1}^{\infty} q_n (t - t_0)^n \ . \tag{5.4}$$

Now, the symbolic expressions of $\mathbf{q_j}$; $\mathbf{j} \geq 2$ are those of Table 1. The values of $\mathbf{q}_0$, $\mathbf{q}_1$, $\boldsymbol{\epsilon}_0$, $\boldsymbol{\lambda}_0$ and $\boldsymbol{\psi}_0$ needed for these expressions could be calculated from the initial values $\vec{\mathbf{r}}_0$ and $\vec{\mathbf{v}}_0$ by the following algorithm .

5.1.1 Computational Algorithm 1

$$1-\ r_0=\left(x_0^2+y_0^2+z_0^2\right)^{1/2}\ ;\ 2-\varepsilon_0=\mu/r_0^3\ ;$$

$$3-\ q_1=\left(x_0\dot{x}_0+y_0\dot{y}_0+z_0\dot{z}_0\right)/r_0\ ;\ 4-\lambda_0=q_1/r_0\ ;$$

$$5-\ \psi_0=\left(\dot{x}_0^2+\dot{y}_0^2+\dot{z}_0^2\right)/r_0^2\ ;\ 6-\ h_x=y_0\dot{z}_0'-z_0\dot{y}_0\ ;$$

$$7-\ h_y=z_0\dot{x}_0-x_0\dot{z}_0\ ;\ 8-\ h_z=x_0\dot{y}_0-y_0\dot{x}_0\ ;$$

$$9-\ q_0=r_0-\left(h_x^2+h_y^2+h_z^2\right)/\mu\ ;\ 10\ -\ END.$$

Finally, it should be mentioned that, the above symbolic expressions are universal in the sense that it could be used for any conic orbits.

5.2. Symbolic Computation of σ

σ is defined by

$$\sigma=\frac{1}{\sqrt{\mu}}\langle\vec{r},\vec{v}\rangle=q\ . \tag{5.5}$$

Differentiating Equation (5.5) with respect to **t** we get

$$\dot{q}=\frac{1}{\sqrt{\mu}}\left\{\left\langle\vec{r},\ddot{\vec{r}}\right\rangle+\langle\vec{v},\vec{v}\rangle\right\}\ , \tag{5.6}$$

$$\ddot{q}=\frac{1}{\sqrt{\mu}}\left\{\left\langle\vec{r},\dddot{\vec{r}}\right\rangle+3\left\langle\vec{v},\ddot{\vec{v}}\right\rangle\right\}\ , \tag{5.7}$$

From Equation (2.1) it follows that

$$\left\langle\vec{r},\ddot{\vec{r}}\right\rangle=-\frac{\mu}{r},\left\langle\vec{r},\dddot{\vec{r}}\right\rangle=\frac{2\mu}{r^3}\langle\vec{r},\vec{v}\rangle\ ;\ \left\langle\vec{v},\ddot{\vec{r}}\right\rangle=-\frac{\mu}{r^3}\langle\vec{r},\vec{v}\rangle\ ,$$

then Equations (5.6) and (5.7) become

$$\dot{q}=\frac{\upsilon^2}{\sqrt{\mu}}-\frac{\sqrt{\mu}}{r}\ , \tag{5.8}$$

$$\ddot{q}=-\in\ q\ . \tag{5.9}$$

Consequently **q** (Equation (5.5)) satisfies the differential Equation (3.1). Then according to Section 3, **σ** could be represented as power series in **t** by

$$\sigma = \sigma_0 + \sum_{n=1}^{\infty} q_n (t - t_0)^n \ , \tag{5.10}$$

where $\boldsymbol{\sigma}_0$ is the value of **σ** at $\mathbf{t} = \mathbf{t}_0$. Again, the symbolic expressions of $\mathbf{q}_0$; $\mathbf{j} \geq 2$ are those of Table 1. The values $\mathbf{q}_0$, $\mathbf{q}_1$, $\in_0$, $\boldsymbol{\lambda}_0$ and $\boldsymbol{\psi}_0$ needed for those expressions could be calculated from the initial values $\vec{\mathbf{r}}_0$ and $\vec{\mathbf{v}}_0$ by the following algorithm.

5.2.1. Computational Algorithm 2

$$1 - \ r_0 = \left(x_0^2 + y_0^2 + z_0^2\right)^{1/2} \ ; \ 2 - \upsilon_0^2 = \dot{x}_0^2 + \dot{y}_0^2 + \dot{z}_0^2 \ ;$$

$$3 - \ H_0 = x_0 \dot{x}_0 + y_0 \dot{y}_0 + z_0 \dot{z}_0 \ ; \ 4 - \in_0 = \mu / r_0^3 \ ;$$

$$5 - \ q_0 = H_0 / \sqrt{\mu} \ \ ; \ \ 6 - \lambda_0 = H_0 / r_0^2 \ ;$$

$$7 - \ q_1 = \frac{\upsilon_0^2}{\sqrt{\mu}} - \frac{\sqrt{\mu}}{r_0} \ ; \ 8 - \psi_0 = \upsilon_0^2 / r_0^2 \ \ ; \ \ 9 - END.$$

The importance of the quantity **σ** is due to its appearance in both, the initial and boundary value problems of space dynamics [see *e.g*. Danby, 1988]. Moreover, **σ** is related to the flight-path angle **Ω** by

$$\sigma = \sqrt{p} \ \tan \ \Omega \ , \tag{5.11}$$

Ω is defined as the angle between the vector $\vec{\mathbf{v}}$ and the local horizontal plane. According to Equations (5.10) and (5.11), **Ω** could then be obtained at any time. This fact is important since, **Ω** is very useful for specifying a satellite's orientation or attitude. This orientation is crucial to determining the effective cross-sectional area required for both drag and solar-radiation perturbations, which may be very important depending on the satellite's mission [Vallado, 1997]. Finally, **σ** is universal expression, that is, it could be used for any type of conic orbits.

5.3. Symbolic Computation of the Eccentric Anomaly

The relation between the eccentric anomaly **E** and time **t** in an elliptic orbit of semi-major axis **a** and eccentricity **e** is the well known Kepler's equation of the form

$$M = \sqrt{\frac{\mu}{\alpha^3}} \ (t - \tau) \ = \ E - e \ \sin \ E \ , \tag{5.12}$$

where **M** is the mean anomaly and **τ** is the time of pericenter passage.

Solving Kepler's equation has intrigued scientists for centuries [Colwell, 1993]. Solutions may be divided into series and iteration methods. In both methods **E** is obtained from the given values of **e** and **M**.

In what follows we shall show that, the solution **E** of Kepler's equation can be obtained as power series in **t** from the given values $\vec{\mathbf{r}}_0$ of and $\vec{\mathbf{v}}_0$.

In elliptic motion we have

$$r = a\ (1 - e\ \mathrm{Cos}\ E)\ , \tag{5.13}$$

$$\sin\ E = \frac{\sigma}{\sqrt{a}}\ , \tag{5.14}$$

where **σ** is given from Equation (5.5) and **α** is given in terms of **r** and **υ** from Equation (2.8) as

$$a = \frac{\mu r}{2\mu - r\upsilon^2}\ . \tag{5.15}$$

Let

$$q\ =\ E\ -\ M\ , \tag{5.16}$$

$$\ddot{q}\ =\ \ddot{E}\ , \tag{5.17}$$

then from Equations (5.12) and (5.13) we get

$$\ddot{E} = -\frac{\mu}{r^3}\ e\ \sin\ E = -\in (E - M) = -\in q\ ,$$

therefore Equation (5.17) becomes

$$\ddot{q}\ =\ -\in\ q\ . \tag{5.18}$$

Consequently, **q** (Equation (5.16)) satisfies the differential Equation(3.1). As in the pervious subsections, **E** could be represented as power series in **t** as

$$E\ =\ \sqrt{\frac{\mu}{a^3}}\ (t - \tau)\ +\ \sum_{n=0}^{\infty} (t - t_0)^n\ . \tag{5.19}$$

As in the above subsection, the symbolic expressions of $\mathbf{q_j}$; $\mathbf{j} \geq 2$ are those of Table 1, while the starting values could also be calculated from the initial values $\vec{\mathbf{r}}_0$ and $\vec{\mathbf{v}}_0$ by the following algorithm

5.3.1. Computational Algorithm 3

$$1-\ r_0=\left(x_0^2+y_0^2+z_0^2\right)^{1/2}\ ;\ 2-\ \upsilon_0^2=\dot{x}_0^2+\dot{y}_0^2+\dot{z}_0^2\ ;$$

$$3-D=x_0\dot{x}_0+y_0\dot{y}_0+z_0\dot{z}_0\ ;\ 4-\sigma_0=D/\sqrt{\mu}\ ;$$

$$5-a=\frac{\mu r_0}{2\mu-r_0\upsilon_0^2}\ ;\ 6-q_0=\tan^{-1}\left\{\frac{\sigma_0}{\sqrt{a}-r_0/\sqrt{a}}\right\}-\sqrt{\frac{\mu}{a^3}}\ (t_0-\tau)\ ;$$

$$7-q_1=\sqrt{\frac{\mu}{a^3}}\left(\frac{a}{r_0}-1\right)\ ;\ 8-\epsilon_0=\mu/r_0^3\ ;$$

$$9-\lambda_0=D/r_0^2\ ;\ 10-\ \psi_0=\upsilon_0^2/r_0^2\ .\ 11-\ END.$$

6. Conclusion

In concluding the present paper, a simple unified approach uses Largrange's fundamental invariants were used to develop time series expansions of four vital two-body problems.

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خوارزمية رمزية موحدة لبعض المفكوكات الزمنية لمسألة الجسمين باستخدام مقادير لاجرانج اللاتغايرية

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المملكة العربية السعودية

المستخلص. تم في هذا البحث تشييد أربعة مفكوكات زمنية لمسألة الجسمين وهي : دوال g, f للاجرانج ، والمسافة القطرية ، وحاصل الضرب القياسي لمتجهي الموضع والسرعة ، وأخيراً الحصة اللامركزية للحركة إلاهيلجية. وللحصول على هذه المفكوكات آنيا وبطريقة موحدة لجميع الصيغ ، فقد اقترضنا أسلوبا موحداً استخدمت فيه مقادير لاجرانج اللاتغايرية لتشييد حل زمني تسلسلي لمعادلة المتذبذب التوافقي.